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ERROR ANALYSIS FOR WINTERS' ADDITIVE

SEASONAL FORECASTING SYSTEM

by

Ed. McKenzie

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ERROR ANALYSIS FOR WINTERS' ADDITIVE SEASONAL FORECASTING SYSTEM

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ABSTRACT

A procedure for deriving the variance of the forecast error for Winters' Additive Seasonal Forecasting system is given. Both point and cumulative T-step ahead forecasts are dealt with. Closed form expressions are given in the cases when the model is (i) trend-free and (ii) non-seasonal. The effects of renormalization of the seasonal factors is also discussed. The fact that the error variance for this system can be infinite is discussed and the relationship of this property with the stability of the system indicated. Some recommendations are given about what to do in these circumstances.

Keywords: Winters Additive Seasonal Model; Forecast Error Variance; Confidence Intervals; System Stability

All forecasts are wrong. As a result it is rarely adequate to give a forecast by itself. Some measure of the reliability or accuracy of the forecast is also required. Such a measure is usually provided by a confidence interval. This is an interval within which the future value we are forecasting will fall with a prescribed probability. The philosophy and construction of such intervals are well documented in a number of forecasting texts and we will not consider them further here. See, for example, Brown (1962), Montgomery and Johnson (1976) and Bowerman and O'Connell (1979).

We simply note that an essential ingredient in all the expressions and formulae involved in such intervals is the variance of the forecast error. Consider the T-step ahead forecast made at time t, i.e. the forecast of X_{t+T} made at time t, $\hat{X}_t(T)$ say. The corresponding error is $e_t(T) = X_{t+T} - \hat{X}_t(T)$. We shall denote its variance, $\mathrm{var}[e_t(T)]$, by σ_T^2. It is clear that σ_T^2 must play a central role in any discussion of the usefulness of $\hat{X}_t(T)$ in predicting X_{t+T}.

The value of σ_T^2 can usually be computed on the assumption that the underlying model is valid. By underlying model we mean the model which the forecasting system assumes is generating the data. The variance can certainly be derived for the ARMA models of Box and Jenkins, and the interested reader is directed to their book (1976). It can also be derived for all the Exponential Smoothing systems discussed by Brown (1962) and others considered more recently by Sweet (1981).

However, no results are available for Winters' seasonal systems. This is particularly unfortunate since these systems are amongst the most commonly used in practice. They are relatively simple to implement and intuitively appealing. Both systems are described in detail in most forecasting texts, e.g. Montgomery and Johnson (1976), Thomopoulos (1980), and Bowerman and O'Connell (1979). In

this last work, approximate confidence intervals are given for Winters' systems. As we shall see, however, these intervals poorly reflect the behaviour of the error variance.

The Winters' Multiplicative seasonal form is a non-linear system and it is difficult to see how any useful information about the forecast error can be obtained directly. Approximation and simulation appear to be the most sensible tools here. In the case of the additive model, however, we can obtain some results.

The purpose of this paper is to give a procedure for deriving the variance of the forecast error for the Winters' Additive Seasonal forecasting system. We deal with not only point forecasts of future values, X_{t+T}, but also cumulative forecasts, i.e. forecasts of $Y_{t,T} = \sum_{i=1}^{T} X_{t+i}$, made at time t.

Expressions are given for the variances of the individual components of the forecasts, i.e. level, trend gradient and seasonal factors, and for the covariances between them. This allows the construction of the variance of any particular forecast and hence of the corresponding forecast error. It also enables us to construct confidence intervals for the components themselves, and linear combinations of them.

Two special cases are considered, viz. when there is no trend, and when there is no seasonality. This latter is the well-known Holt-Winters non-seasonal forecasting system. In both cases, closed expressions are obtained for the variances. In the completely general seasonal case, the expressions derived involve a number of unknowns which are obtained by solving a set of linear equations.

Some necessary discussion is also given about the stability of these forecasting systems.

1. THE RESULTS

The underlying model assumed by the Winters' additive seasonal system is given by

$$X_t = m + bt + s_k + a_t \tag{1}$$

where $t = rn + k$, and n is the length of the season. The values $\{s_k : k = 1,2,...,n\}$ are the additive seasonal factors and $\{a_t\}$ is a sequence of independent identically distributed random variates of zero mean and variance σ^2.

The standard form of the corresponding forecasting system is given by

$$\begin{aligned} m_t &= \alpha_0(X_t - S_{t-n}) + (1-\alpha_0)(m_{t-1} + b_{t-1}) \\ b_t &= \alpha_1(m_t - m_{t-1}) + (1-\alpha_1)b_{t-1} \\ S_t &= \alpha_2(X_t - m_t) + (1-\alpha_2)S_{t-n} . \end{aligned} \tag{2}$$

The T-step ahead forecast is

$$\hat{X}_t(T) = m_t + Tb_t + S_{t+k-n} \tag{3}$$

where $T = rn + k$, $(k = 1,2,...,n;\ r \geq 0)$.

The cumulative forecast, i.e. the forecast of

$$Y_{t,T} = \sum_{i=1}^{T} X_{t+i} \text{ is given by } \hat{Y}_{t,T} = \sum_{i=1}^{T} \hat{X}_t(i) .$$

Our derivation of the forecast error variances parallels the development given for Exponential Smoothing models by Brown (1962). By definition, $\hat{X}_t(T)$ depends on only past values of a_t and so is independent of X_{t+T}. Thus, $\sigma_T^2 = \sigma^2 + \mathrm{var}[\hat{X}_t(T)]$. Further, $\hat{X}_t(T)$ can be expressed as a linear combination of these past values of a_t and so its variance is proportional to σ^2. Thus,

$\sigma_T^2 = \sigma^2[1 + V(T)]$. When dealing with cumulative forecasts, the same argument holds. The error now is $E_{t,T} = Y_{t,T} - \hat{Y}_{t,T}$ and we may show it has variance of the form $\sigma^2[T + V_E(T)]$.

In practice, σ^2 is unknown. It may be estimated from the data, however, using the fact that $\sigma^2[1 + V(1)] = \sigma_1^2$, the variance of the one step ahead forecast error. Thus, we can estimate σ_1^2 directly from the forecast errors, and knowledge of $V(1)$ yields an estimate of σ^2 .

We derive $V(T)$ and $V_E(T)$ from the variance-covariance matrix of the estimates of the individual components of the forecast i.e. m_t, b_t and S_{t+k-n}, $k = 1,2,\dots,n$. These variances are of some interest in their own right for the construction of confidence intervals for the components, and we display them below. Details of the derivations are given in Appendix 1.

$$\sigma_m^2 = \mathrm{Var}(m_t) = \alpha_0^2\{\alpha_1^2[nd_0 + 2\sum_{i=1}^{n-1}(n-i)d_i] + 2(1-\alpha_1)(d_0-d_n)\}\sigma^2$$

$$\sigma_b^2 = \mathrm{Var}(b_t) = 2\alpha_0^2\alpha_1^2(d_0-d_n)\sigma^2$$

$$\sigma_s^2 = \mathrm{Var}(S_{t+k-n}) = 2\alpha_2^2(1-\alpha_0)^2(d_0-d_1)\sigma^2, \quad k = 1,2,\dots,n$$

$$\sigma_{mb} = \mathrm{Cov}(m_t,b_t) = \alpha_0^2\alpha_1(2-\alpha_1)(d_0-d_n)\sigma^2$$

$$\sigma_{bk} = \mathrm{Cov}(b_t,S_{t+k-n}) = \alpha_0\alpha_1\alpha_2(1-\alpha_0)(d_{n-k}-d_k + d_{k-1} - d_{n-k+1})\sigma^2, \quad k = 1,2,\dots,n .$$

$$\sigma_{mk} = \mathrm{Cov}(m_t,S_{t+k-n}) = \alpha_0\alpha_2(1-\alpha_0)[(1-\alpha_1)(d_{n-k}-d_k) + (d_{k-1}-d_{n-k+1})]\sigma^2, \quad k = 1,2,\dots,n .$$

$$\sigma_{ij} = \sigma_{ji} = \mathrm{Cov}(S_{t+i-n},S_{t+j-n}) = \alpha_2^2(1-\alpha_0)^2(2d_{j-i} - d_{j-i-1} - d_{j-i+1})\sigma^2, \quad 1 \le i < j \le n.$$

Before discussing the derivation of the values $\{d_0,d_1,\dots,d_n\}$ we note the forms of $V(T)$ and $V_E(T)$.

Now, $V(T)\sigma^2 = \text{var}(m_t + Tb_t + S_{t+k-n})$

$$= \sigma_m^2 + T^2\sigma_b^2 + \sigma_s^2 + 2(\sigma_{mk} + T\sigma_{mb} + T\sigma_{bk}) .$$

Thus,

$$\begin{aligned}
V(T) = {} & 2\alpha_0^2\alpha_1^2(d_0-d_n)T^2 \\
& + 2\alpha_0\alpha_1[\alpha_0(2-\alpha_1)(d_0-d_n) + \alpha_2(1-\alpha_0)(d_{n-k} - d_k + d_{k-1} - d_{n-k+1})]T \\
& + 2\alpha_0\alpha_2(1-\alpha_0)[(1-\alpha_1)(d_{n-k}-d_k) + (d_{k-1}-d_{n-k+1})] \\
& + 2\alpha_0^2(1-\alpha_1)(d_0-d_n) + 2\alpha_2^2(1-\alpha_0)^2(d_0-d_1) + \alpha_0^2\alpha_1^2[nd_0 + 2\sum_{i=1}^{n-1}(n-i)d_i]
\end{aligned} \tag{4}$$

In the same way, using the fact that

$$\hat{Y}_{t,T} = Tm_t + \frac{1}{2}T(T+1)b_t + \sum_{k=1}^{T} S_{t+k-n} ,$$

we can show that when $T = pn + q,\ p \geq 0,\ 1 \leq q \leq n$:

$$\begin{aligned}
V_E(T) = {} & [T^2\sigma_m^2 + T^2(T+1)\sigma_{mb} + \frac{1}{4}T^2(T+1)^2\sigma_b^2]/\sigma^2 \\
& + 2\alpha_2^2(1-\alpha_0)^2[(p+1)^2(d_0-d_q) + p^2(d_0-d_{n-q}) + p(p+1)(d_q-d_n+d_{n-q}-d_0)] \\
& + \alpha_0\alpha_1\alpha_2(1-\alpha_0)[(2p+1)(d_0-d_n) + (d_{n-q}-d_q)]T(T+1) \\
& + 2\alpha_0\alpha_2(1-\alpha_0)[p(2-\alpha_1)(d_0-d_n) + (d_0-d_q-d_n+d_{n-q}) \\
& + \alpha_1\sum_{k=1}^{q}(d_k-d_{n-k})]T
\end{aligned} \tag{5}$$

Now the numbers $d_0, d_1, \ldots, d_n$ are necessary to evaluate the variances. These $(n+1)$ values are derived by means of an algorithm which is given and discussed in Appendix 2. It is a fairly straightforward algorithm due to Wilson (1979). It has the property that it also checks the stability of the forecasting system. As we shall see in Section 3 this is most useful for this

forecasting system. Before discussing any of the technical aspects of the procedure described here, we present some particular cases which have closed-form solutions.

2. SPECIAL CASES

2.1. Trend-free additive seasonal model.

In this case the model assumed is a constant level plus additive seasonals. The forecasting equations are obtained by removing the second equation from the original three (2), and dropping b_{t-1} from the first. Here the values $d_0, d_1, \ldots, d_n$ can be derived algebraically. The variance-covariance matrix of the components is now obtained from

$$\sigma_m^2 = n\alpha_0^2 h \,,$$
$$\sigma_s^2 = \alpha_2(1-\alpha_0)[(n-1)\alpha_0 + \alpha_2(1-\alpha_0)]h \,,$$
$$\sigma_{mk} = \alpha_0\alpha_2(1-\alpha_0)h \,, \qquad 1 \le k \le n \,,$$
$$\sigma_{ij} = \sigma_{ji} = - \sigma_{mk} \,, \qquad 1 \le i < j \le n \,,$$

where $h = \sigma^2/[(2-\alpha_0-\alpha_2+\alpha_0\alpha_2)(n\alpha_0+\alpha_2-\alpha_0\alpha_2)]$.

Thus,

$$V(T) = \frac{1 - (1-\alpha_0)(1-\alpha_2)}{1 + (1-\alpha_0)(1-\alpha_2)} \,. \tag{6}$$

It is interesting to note that for this model $V(T)$ is independent of T .

For the cumulative forecast error, we find that for $T = pn + q$, $(p \ge 0,$ $1 \le q \le n)$, $V_E(T) = PT^2 + \alpha_2^2(1-\alpha_0)^2(pT + pq + q)$ where $P = \alpha_0[n\alpha_0 + \alpha_2(1-\alpha_0)]h/\sigma^2$

2.2. Non-seasonal Holt-Winters' system.

The appropriate equations are obtained by removing the third equation from (2) and S_{t-n} from the first. This is a well-known non-seasonal forecasting system and often used for linear trend models. It is a direct competitor of Brown's second order, or double, exponential smoothing which we will consider shortly. The underlying model is assumed to be a simple linear trend i.e. s_k is removed from (1).

We obtain

$$\sigma_m^2 = (2\alpha_1 + 2\alpha_0 - 3\alpha_0\alpha_1)h_1 ,$$

$$\sigma_b^2 = 2\alpha_0\alpha_1^2 h_1 ,$$

$$\sigma_{mb} = \alpha_0\alpha_1(2-\alpha_1)h_1 ,$$

where

$$h_1 = \sigma^2/(4-2\alpha_0-\alpha_0\alpha_1) .$$

We can deduce in the usual way that

$$V(T) = [2\alpha_0\alpha_1^2 T^2 + 2\alpha_0\alpha_1(2-\alpha_1)T + (2\alpha_0+2\alpha_1-3\alpha_0\alpha_1)]h_1/\sigma^2 \tag{7}$$

and

$$V_E(T) = [\alpha_0\alpha_1^2 T^2 + 4\alpha_0\alpha_1 T + (4\alpha_0+4\alpha_1-2\alpha_0\alpha_1-\alpha_0\alpha_1^2)]T^2 h_1/2\sigma^2$$

2.3. Brown's Double Exponential Smoothing.

It is well-known that this system is equivalent to the Holt-Winters' non-seasonal form above if we choose $\alpha_0 = 1-\beta^2$ and $\alpha_1 = (1-\beta)/(1+\beta)$ where $\beta = 1-\alpha$ and α is the smoothing constant. Thus, the corresponding results can be obtained by making these substitutions in the equations of Section 2.2. above. The results for σ_m^2, σ_b^2, σ_{mb} and $V(T)$ may be found in Brown (1962) and $V_E(T)$ is given in Bowerman and O'Connell (1979).

2.4. Continuously re-normalized seasonal factors.

In the general additive seasonal model given by (1) it is usually assumed that $\sum_{k=1}^{n} s_k = 0$. This is done to ensure some measure of independence between the level of the data and the seasonal pattern. It is always recommended that when the corresponding forecasting system, as given by (2), is implemented, the seasonal factors sum to zero initially. What is to be done

thereafter is not so clear. Because of the revision equation for the seasonal factors they will no longer sum to zero after the first observation. We can renormalize the seasonal factors at any time by subtracting from each the average of the set, i.e. the most recent n. Whether we should do so or not is not clear. Some authors seem to recommend against it, e.g. Bowerman and O'Connell (1979); some regard it as an optional modification, e.g. Montgomery and Johnson (1976); some recommend renormalization once per season, e.g. Chatfield (1978); and some recommend continuous renormalization, i.e. after every revision of seasonal factors, e.g. Thomopoulos (1980).

Our only interest here is in how such a procedure affects the forecast error variance. In the cases of occasional or purely seasonal renormalization the situation is very complex and we have nothing to say, except that the effect on the error variance appears to be small. In the case of continuous renormalization, however, the following result may be applied. An outline of the proof is given in Appendix 1. Continuous renormalization of the general forecasting system given by (2) yields exactly the same forecasts (and so forecast errors) as running the system without renormalization but replacing $\alpha_0, \alpha_1, \alpha_2$ by $\alpha_0^*, \alpha_1^*\ \alpha_2^*$ respectively, where $\alpha_0^* = \alpha_0 - \alpha_2(1-\alpha_0)/n$, $\alpha_1^* = \alpha_0\alpha_1/\alpha_0^*$, and $\alpha_2^* = \alpha_2/(1+\alpha_2/n)$. As a consequence, the appropriate covariance matrix elements and values of $V(T)$ and $V_E(T)$ can be obtained as in Section 1 above by replacing $\alpha_0, \alpha_1, \alpha_2$ by $\alpha_0^*, \alpha_1^*, \alpha_2^*$ respectively. The same holds true for the trend-free version discussed in Section 2.1.

3. TECHNICAL CONSIDERATIONS

3.1. Infinite Error Variance and Stability.

In smoothing systems in general the variance of the forecast error increases as the values of the smoothing constants increase. For example, in the case of the simple exponentially weighted moving average (SEWMA), $\hat{X}_t = \alpha X_t + (1-\alpha)\hat{X}_{t-1}$, the variance of the forecast error is proportional to $2/(2-\alpha)$ which clearly increases with α in $(0,1)$. The need for higher values of α in practice reflects the fact that the underlying level is changing rapidly. Consequently, a more responsive forecast is needed. Moreover, the inherent instability of the underlying model is reflected in the increase in forecast error variance. This variance is finite while α remains in the stability region of the system, i.e. $(0,2)$ for the SEWMA. Note that the set of values from which α is usually chosen is a subset of the stability region of the system. Thus, a stable system always results for the SEWMA. This is also true for all the General Exponential Smoothing models. However, it is not true for the seasonal system under consideration here.

It is a somewhat surprising and problematic fact that there are choices of the smoothing constants α_0, α_1, α_2 lying in the usual range, $(0,1)$, which yield an infinite variance for the forecast error. The algorithm given in Appendix 2 checks for this possibility which indicates that the forecasting system is unstable. If such a situation arises clearly no meaningful confidence intervals can be constructed. More importantly, however, we must decide how to interpret this knowledge of the system's instability. The concept of stability for a system of difference equations is an important one but rarely discussed in the context of forecasting systems. Two useful exceptions to this are the papers by McClain (1974) and Brenner et al, (1968).

In essence, a forecasting system such as (2) above is stable if the influence of earlier observations decreases with the passage of time. Thus, the forecasts, or (equivalently) the forecast errors, are influenced more by recent observations than by those in the past. It is worth noting that this is also the essential philosophy of exponential smoothing systems, and indeed, most forecasting systems.

The converse of this is that in an unstable system past observations have a constant or even growing influence on future forecasts. As an illustration, consider the SEWMA again. The forecast $\hat{X}_t$ can be written as a weighted average of all past observations. At time t, the weight given to X_{t-k} is $\alpha(1-\alpha)^k$. Clearly, if α lies outside the stability region (0,2) this weight increases with k so that data in the most remote part have greatest influence upon the forecast. Equally, if $\alpha = 2$, all observations, however distant in time, make the same contribution to the forecast.

The inescapable conclusion is that it would be extremely unwise to select smoothing constants which do not lie in the stability region of the system. On the one hand, we can tolerate a high (but finite) forecast error variance because this represents a trade-off between accuracy and robustness. In an effort to predict a model whose parameters are changing rapidly in time we may require a more responsive system. The cost of this is a correspondingly higher error variance. On the other hand, we can not tolerate an infinite error variance because this indicates an unstable forecasting system. Such a system violates exactly those assumptions which we hold most important to the generation of our forecasts.

3.2. Practical consequences.

It is obvious from the forms of $V(T)$ in the two special cases (6) and (7) that the usual range of values of smoothing constants, i.e. (0,1), lies

within the stability regions for these systems. The problem arises for only the general seasonal model. The actual stability region is difficult to establish in general here for it depends not only on α_0, α_1 and α_2 but also on n, the length of the season. However, Gardner (1984) reports that Sweet has demonstrated numerically that the usual range for the three smoothing constants lies within the stability region for n up to four. For seasons longer than four periods it is no longer true: a result of importance for weekly and monthly data.

The values of the smoothing constants are established in one of two ways in general. They may be selected intuitively by appealing to the ideas of required speed of response and constancy of the underlying model. Alternatively, they may be chosen as giving the best fit with respect to some criterion such as least squares. However the values are chosen, we should be very concerned if they do not lie in the systems's stability region.

If we have chosen them for rapid response we may allow their use for a brief period, perhaps when we initialize the forecasting system. As noted above, however, if we do not replace them with values in the stability region then the longer we forecast the more influential become the very first observations. If some best fitting criterion selects 'unstable' smoothing constants, i.e. ones leading to an infinite error variance, then a very real possibility is that the model is wrong, at least for part of the data. In particular, it may well be that a multiplicative seasonal model is called for.

We may decide to retain the model but use smoothing constants in the stability region. These will be obtained by decreasing one or more of α_0, α_1, α_2. In general, if α_0, α_1, α_2 do not lie in the stability region it will be because one or more of them is too large.

4. SUMMARY

The purpose of this paper has been to give a procedure for deriving the variance of the forecast error for Winters' Additive Seasonal System. In summary that procedure is as follows:

(i) If the model has no trend or is non-seasonal the results are given in Section 2.1 and 2.2 respectively in closed form.

(ii) From the chosen smoothing constants α_0, α_1 and α_2 the coefficients $W_1, W_2, \ldots, W_{n+1}$ are derived as in Appendix 1.

(iii) The W_i values are used to start the algorithm of Appendix 2 and yield $d_0, d_1, \ldots, d_n$. The algorithm simultaneously checks the stability of the forecasting system for the chosen values α_0, α_1, α_2. If $n \leq 4$ the system will be stable.

(iv) The values of d_i are substituted into the appropriate expressions for $V(T)$ and $V_E(T)$.

Note that at step (iv) we are able to evaluate the variance of the errors for T-step ahead forecasts and T-step ahead cumulative forecasts. Confidence intervals for these future values can be derived from these variances.

A discussion of how to calculate the corresponding results when the seasonal factors are renormalized after each observation is given. We have also discussed the (real) possibility that the error variance may be infinite and indicated its relationship with the stability of the forecasting system. Some recommendations are made about the interpretation of system instability.

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APPENDIX 1

The results of this paper are obtained using the fact that the forecasting system given by (2) has an equivalent ARIMA process. The nature of this equivalence is discussed in McKenzie (1984) and the equivalent ARIMA for (2) is given in McKenzie (1976). It has the form $(1-B)(1-B^n)X_t = W(B)e_t$, where B is the backshift operator defined by $B^k X_t = X_{t-k}$ and $W(B) = 1 - \sum_{k=1}^{n+1} W_k B^k$ where $W_1 = 1 - \alpha_0 - \alpha_0\alpha_1$, $W_n = 1 - \alpha_0\alpha_1 - \alpha_2(1-\alpha_0)$, $W_{n+1} = -(1-\alpha_0)(1-\alpha_2)$, and $W_k = -\alpha_0\alpha_1$ for $k = 2,3,\ldots,n-1$.

Suppose now that $W(B)\delta(B) = 1$ where $\delta(B) = \sum_{k=0}^{\infty} \delta_k B^k$. Using equations (2), it can be shown that, for the purposes of the variance calculation, the components of of the forecast can be expressed as infinite moving averages in $\{a_t\}$. Thus:

$$m_t = \alpha_0\{1 + \sum_{i=1}^{\infty} [\zeta_i - (1-\alpha_1)\zeta_{i-1}]B^i\}a_t ,$$

$$b_t = \alpha_0\alpha_1[\sum_{i=0}^{n-1} \delta_i B^i + \sum_{i=n}^{\infty} (\delta_i - \delta_{i-n})B^i]a_t ,$$

$$S_t = \alpha_2(1-\alpha_0)[1 + \sum_{i=1}^{\infty} (\delta_i - \delta_{i-1})B^i]a_t ,$$

where $\zeta_k = \sum_{i=0}^{\min(k,n-1)} \delta_{k-i}$. Further, the corresponding representation of S_{t+k-n} can be obtained by writing it as $B^{n-k}S_t$, $(k = 1,2,\ldots,n)$.

Since $\{a_t\}$ are independent random variables the variances and covariances can be obtained directly from these moving average representations. Defining $d_k = \sum_{i=0}^{\infty} \delta_i\delta_{i+k}$, $(k = 0,1,\ldots,n)$, yields the expressions given in the paper. The algebra is tedious and hardly illuminating and so is omitted. It can be obtained upon request from the author.

As regards the renormalization procedure, recall the equivalent ARIMA for (2) given above. If we introduce continuous renormalization as discussed we find there is still an equivalent ARIMA and it has the same form. Now, however, W(B) has α_0, α_1, α_2 replaced by α_0^*, α_1^*, α_2^*. This ARIMA is equivalent to the system (2) with the starred smoothing constants. Hence, the result. Initially, the result is surprising. However, note that we are really dealing with two different decompositions of the seasonal factor. One form is (m_t+S_{t+k-n}) as it appears in (2) and the other is $(m_t^*+S_{t+k-n}^*)$ where S_{t+k-n}^* is normalized to sum to zero and m_t^* is the correspondingly adjusted level.

APPENDIX 2

To obtain the values $\{d_k\}$ note that, by definition, they are the variance (d_0) and the first n autocovariances $(d_1, d_2, \ldots, d_n)$ of the autoregressive process of order $(n+1)$ given by $W(B)Z_t = \varepsilon_t$, where $\{\varepsilon_t\}$ are independent random variables of zero mean and unit variance. Thus, the sequence $\{d_k\}$ may be obtained by solving a suitable set of the Yule-Walker equations for this process. This procedure is discussed in McLeod (1975, 1977).

From our point of view, however, a much superior approach is presented by Wilson (1979). The stability of the forecasting system corresponds to the stationarity of the autoregressive process $\{Z_t\}$ and can be tested routinely within the procedure. The algorithm is as follows:

(i) define $W_{n+1,k} = W_k$, $k = 1,2,\ldots,n+1$; and $t_{n+1} = 1$.

(ii) apply the following equations in the given order for $k = n+1, n, \ldots, 2$

$$D_k = 1 - W_{k,k}^2$$

if $D_k \leq 0$, the system is unstable; stop now.

$$W_{k-1,i} = (W_{k,i} + W_{k,k} W_{k,k-i})/D_k, \quad (i = 1,2,\ldots,k-1)$$

$$t_{k-1} = t_k/D_k$$

(iii) $D_1 = 1 - W_{1,1}^2$

if $D_1 \leq 0$, the system is unstable; stop now.

$d_0 = t_1/D_1$, and the system is stable, and the error variance finite.

(iv) $d_1 = W_{1,1}\, d_0$

$$d_k = \sum_{i=1}^{k-1} W_{k-1,i}\, d_{k-i} + W_{k,k}\, t_k, \quad k = 2,3,\ldots,n$$

Note that if $n \leq 4$ the system is stable for the usual choice of parameters i.e. $\alpha_i \in (0,1)$, $i = 0,1,2$, and so we need not test D_k.

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